Non-linear vibrations and instabilities of orthotropic cylindrical shells with internal flowing fluid

Zenon del Prado a, Paulo B. Gonçalves b,*, Michael P. Païdoussis c

a Civil Engineering Department, UFU, 74605-220, Goiânia, GO, Brazil
b Civil Engineering Department, Catholic University, PUC-Rio, 22451-900—Rio de Janeiro, RJ, Brazil
c Department of Mechanical Engineering, McGill University, 817 Sherbrooke Street W., Montréal, Québec, Canada H3A 2K6

ARTICLE INFO

Article history:
Received 10 September 2009
Received in revised form 13 March 2010
Accepted 25 March 2010
Available online 31 March 2010

Keywords:
Cylindrical shells
Fluid–structure interaction
Non-linear vibrations
Orthotropic material

ABSTRACT

In this work, Donnell’s non-linear shallow shell equations are used to study the dynamic instability of perfect simply supported orthotropic cylindrical shells with internal flowing fluid and subjected to either a compressive axial static pre-load plus a harmonic axial load or a harmonic lateral pressure. The fluid is assumed to be non-viscous and incompressible and the flow, isentropic and irrotational. An expansion with eight degrees of freedom, containing the fundamental, companion, gyroscopic, and four axi-symmetric modes is used to describe the lateral displacement of the shell. The Galerkin method is used to obtain the non-linear equations of motion which are solved by the Runge–Kutta method. A detailed parametric analysis clarifies the influence of the orthotropic material properties on the non-linear buckling and vibration characteristics of the shell. Numerical methods are used to identify the effect of the fluid flow and applied loads control parameters on the bifurcations and stability of the shell motions.

© 2010 Published by Elsevier Ltd.

1. Introduction

Cylindrical shells are widely used structures in several engineering areas for conveying fluid, being one of the most common shell geometries in industrial applications. The buckling and vibration analysis of cylindrical shells under various loading conditions has thus become an important research area in applied mechanics. Also, the analysis of fluid–shell interaction has been a topic of continuous interest in the last decades. However, most of these investigations are concerned with the analysis of elastic isotropic shells in contact with internal and external quiescent or flowing fluids.

A reduced number of these works has as object of investigation, the behavior of orthotropic shells. Among the previous studies related to the present investigation, we can mention the work of Jain [1] who studies the free vibrations of orthotropic cylindrical shells partially or completely filled with an incompressible, non-viscous fluid; Warburton and Soni [2] study the resonant response of orthotropic cylindrical shells, while Bradford and Dong [3] investigate the lateral vibrations of orthotropic cylinders under initial stress.

Ip et al. [4], using Love’s first-approximation shell theory, describe a procedure to model the free vibration responses of fiber-reinforced composite cylindrical shells in a free–free configuration. A set of linear equations of motion is derived using the Rayleigh–Ritz method where the shell vibration mode shapes are described by characteristic beam functions. With that model, inextensional Rayleigh and Love modes can be identified having frequencies close to each other. The contributions to the strain energy due to various elastic properties are also investigated. They show that, when increasing the shell thickness, the circumferential modulus provides a major portion of the flexural energy of the vibrating structure; while the longitudinal and in-plane shear moduli contribute mostly to the stretching energy; reducing the shell thickness results in a substantial increase in the ratio of the energies associated with the longitudinal and shear moduli.

Chen et al. [5] study the free vibrations of orthotropic cylindrical shells based on the three-dimensional elasticity theory considering the effect of internal fluid. They obtain the frequency equation of non-axisymmetric free vibration modes of an orthotropic fluid-filled cylindrical shell with arbitrary constant thickness and compare their results with those based on shell theory, and then Chen and Ding [6] analyze the free vibrations of fluid-filled transversely isotropic cylindrical shells. A shell is called transversely isotropic if the in-plane properties are different from those in the perpendicular direction.

Using the Sanders–Koiter non-linear shell theory, Selmane and Lakis [7] study the influence of non-linearities associated with the shell wall and with the fluid flow on the dynamics of elastic thin orthotropic cylindrical shells. They consider non-uniform open
submerged cylindrical shells subjected simultaneously to an internal and external fluid flow by using a hybrid finite element method. Results show the influence of non-linearities on the free vibrations of totally submerged open or closed cylindrical shells; both softening and hardening behavior of the shell are observed.

Lakis et al. [8] present a hybrid method to predict the influence of geometric non-linearities on the natural frequencies of an empty laminated orthotropic cylindrical shell. The Sanders–Koiter non-linear strain–displacement relations are used to formulate the shell equations in combination with the finite element method. Results show the influence of axial and circumferential half-waves on the non-linear frequencies of the shell. The analyzed shells display a hardening behavior.

Mao and Williams [9], using a non-linear theory for non-shallow shells, analyze the parametric resonance of orthotropic circular cylindrical shells under harmonically varying axial compression. In the analysis, the transverse shear deformation is taken into account by considering a first-order shell theory. Numerical results show the dependence of the post-critical behavior on the properties of the material, geometry and excitation parameters.

Li and Chen [10], using Flügge’s linear shell theory, study the dynamic response of orthotropic circular cylindrical shells subjected to external hydrostatic pressure. To analyze the dynamic response of the shell, the normal mode theory is used. They investigate in detail the effect of shell parameters, external hydrostatic pressure and material properties on the dynamic behavior of the shell.

Recently, Mallon [11] analyzes experimentally and numerically a base-excited thin orthotropic cylindrical shell with a top mass. The shell is made of Poly-ethylene Terephthalate and the orthotropy is due to the fabrication process [12]. Daneshjou et al. [13] conduct an analytical study to understand the characteristic of sound transmission through an orthotropic cylindrical shell with subsonic external flow.

Apart from naturally orthotropic materials, several shell problems can be effectively analyzed using an orthotropic shell theory. For example, the stability and vibration analysis of densely ring- and string-stiffened cylindrical shells, such as those used in aero-space engineering structures, are usually accomplished by replacing the stiffened structure by an equivalent orthotropic continuum and the effect of stiffeners is averaged or “smeared out” over the shell [14]. Through this method, when the wavelength of vibration is much larger than the distance between stiffeners, very accurate results are achieved. Shen [15] studies the post-buckling analysis of stiffened laminated cylindrical under combined external liquid pressure and axial compression, the ‘smeared stiffener’ approach is adopted for the stiffeners. The analysis uses a singular perturbation technique to determine the interactive buckling loads and post-buckling equilibrium paths. Torkamani et al. [16] investigate experimentally and numerically the free vibrations of orthogonally stiffened cylindrical shells using the similitude theory. The Donnell-type non-linear strain–displacement relations along with the smeared theory are used to model the structure.

The analysis of corrugated shells (plates) can also be analyzed as thin, equivalent orthotropic shells of uniform thickness. Briassoulis [17] derived equivalent orthotropic properties of corrugated sheets and reviewed some analytical expressions for the equivalent rigidities of orthotropic thin shells given in the literature. Using this approach, Wang et al. [18] study the non-linear free vibrations of corrugated circular plates, obtaining the analytical solutions for the amplitude–frequency relationship through a perturbation method. Liu and Li [19] study the non-linear bending and free vibration for corrugated circular plate via Galerkin’s and a modified iteration method. Larbi [20] studies the buckling of corrugated tin cans under uniform external pressure modeled as thin-walled orthotropic cylindrical shells. Finally, many problems in biology can also be investigated using the orthotropic shell theory [21].

In this work, the non-linear vibrations of a simply supported fluid-filled orthotropic cylindrical shell subjected to axial time-dependent loads and lateral harmonic pressure are analyzed. To model the shell, the Donnell non-linear shallow shell theory without considering the effect of shear deformation is used. The fluid is assumed to be incompressible and inviscid and the flow to be isentropic and irrotational. The incompressibility hypothesis is true for liquid-filled shells vibrating in the low-frequency range. A model with eight degrees of freedom, satisfying the relevant boundary and continuity conditions, and containing the fundamental, companion, gyroscopic, and four axi-symmetric modes, is used to describe the lateral displacements of the shell and the Galerkin method is applied to derive a set of coupled non-linear ordinary differential equations of motion which are, in turn, solved by the Runge–Kutta method. The statically pre-loaded shell is considered to be initially at rest, in a position corresponding to a pre-buckling configuration. The results clarify the marked influence of the orthotropic material properties, fluid flow and load parameters on the dynamic stability boundaries and bifurcations. To the authors’ best knowledge, no such detailed analysis of the influence of material orthotropy on the non-linear buckling and dynamic behavior of cylindrical shells with fluid flow can be found in literature. The basic theory backing the present investigation can be found in Paidoussis [22] and Amabili [23].

2. Mathematical formulation

2.1. Shell equations

Consider a thin-walled simply supported cylindrical shell with radius R, thickness h, length L, containing an internal flowing fluid and subjected to either a harmonic lateral pressure or a time-dependent axial load. The middle surface of the shell is defined as the reference surface. The axial, circumferential and radial coordinates are denoted by x, y = Rθ, and z, respectively, and the corresponding displacements of the shell middle surface are denoted by u, v, and w, as shown in Fig. 1. It is assumed that the local coordinate system, which determines the principal axes of material orthotropy, coincides with the global cylindrical coordinates. The shell is made of an elastic orthotropic material with Young’s moduli $E_{xx}$ and $E_{yy}$ in the axial and circumferential directions, respectively, shear modulus $G_{xy}$, Poisson coefficients $v_{xx}$ and $v_{yy}$, and mass density $\rho_v$. In this work the mathematical formulation follows that previously presented in Refs. [24–27].

For an orthotropic material, obeying the generalized Hooke’s law, the stress resultant–strain relations are given by

$$
\begin{bmatrix}
N_{x0} \\
N_{y0} \\
N_{z0} \\
M_{xx} \\
M_{yy} \\
M_{zz}
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & 0 & 0 & 0 & 0 \\
A_{21} & A_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & A_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & D_{11} & D_{12} & 0 \\
0 & 0 & 0 & D_{21} & D_{22} & 0 \\
0 & 0 & 0 & 0 & D_{33} & 0
\end{bmatrix} \begin{bmatrix}
e_{xx} \\
e_{yy} \\
e_{zz} \\
e_{xy} \\
e_{yx} \\
e_{zz}
\end{bmatrix}
$$

(1)

where $N_{x0}$, $N_{y0}$, and $N_{z0}$ are the in-plane normal and shearing force intensities per unit length along the edge of a shell element; $M_{xx}$, $M_{yy}$, and $M_{zz}$ the bending and twisting moment resultants; $e_{xx}$ and $e_{yy}$ the extensional strain in the axial and circumferential directions and $e_{xy}$ the shearing strain components at a point on the shell middle surface; $K_{xx}$ and $K_{yy}$ the curvature changes and
\[ f = f_0 \sin (\pi \zeta/L) \cos (n\theta) \cos (o_l t) \]

**Fig. 1.** Shell characteristics. (a) Shell geometry, (b) axial harmonic load and fluid flow, and (c) harmonic lateral pressure.

The compatibility equation is given by

\[ P_{22} \frac{\partial^4 F}{\partial \zeta^4} + \frac{1}{R} (P_{33} - 2P_{12}) \frac{\partial^2 F}{\partial \zeta^2 \theta^2} + \frac{1}{R^2} P_{11} \frac{\partial^4 F}{\partial \theta^4} = \frac{1}{R} \frac{\partial^2 W}{\partial \zeta^2} + 4 \left( \frac{\partial^2 W}{\partial \zeta^2} \right)^2 - \frac{\partial^2 W}{\partial \zeta^2} \frac{\partial^2 W}{\partial \theta^2} \]  

where

\[ P_{11} = \frac{A_{12}}{A_{11} A_{22} - A_{12}^2}, \quad P_{12} = \frac{A_{12}}{A_{11} A_{22} - A_{12}^2}, \quad P_{22} = \frac{1}{A_{13}}. \]  

The simply supported boundary conditions for the shell with axial load at \( x = 0 \) and \( x = L \) are

\[ w = 0, \]  

\[ M_{xx} = \left[ D_{11} \frac{\partial^2 W}{\partial \zeta^2} + D_{12} \left( \frac{1}{R^2} \frac{\partial^2 W}{\partial \theta^2} \right) \right] = 0, \]  

\[ N_{xx} = \tilde{N}_x(t), \quad v = 0. \]  

The uniformly distributed axial load along the edges \( x = 0 \) and \( x = L \) is given by

\[ \tilde{N}_x(t) = -\frac{P_x}{2\pi R} \frac{P_d \cos(o_l t)}{2\pi R}. \]  

where \( P_x \) is a compressive uniform static load, \( P_d \) the magnitude of the harmonic axial load, \( t \) the time, and \( o_l \) the forcing angular frequency.

### 2.2. Expansion for the lateral displacement

In this work, the following modal expansion for the lateral displacements \( w(x, \theta, t) \) in terms of the circumferential and axial
variables is adopted [25]
\[
\begin{align*}
  & w(x,t) = \xi_{1,1}(t) \sin q \cos(n \pi x) + \xi_{1,2}(t) \sin q \sin(n \pi x) \\
  & + \xi_{2,1}(t) \sin 2q \cos(n \pi x) + \xi_{2,2}(t) \sin 2q \sin(n \pi x) \\
  & + \xi_{3,1}(t) \sin q \cos(3 \pi x) + \xi_{3,3}(t) \sin q \sin(3 \pi x) \\
  & + \xi_{4,5}(t) \sin 5q + \xi_{4,7}(t) \sin 7q,
\end{align*}
\]
(13)

where \( \xi_{j,k}(t) \) and \( \xi_{j,k}(t) \) are the time-dependent non-dimensional modal amplitudes, \( q = m \pi x \) and \( m \) and \( n \) are, respectively, the number of half-waves in the axial direction and the number of waves in the circumferential direction. This modal expansion satisfies the out-of-plane boundary conditions (9) and (10) and includes the basic vibration mode, the companion mode, gyroscopic modes and four axi-symmetric modes. The choice of these modes is based on previous investigations on modal solutions for the non-linear analysis of cylindrical shells [27,30–32]. They show that these modes take into account all relevant non-linear modal interactions observed in the past in the non-linear vibrations of cylindrical shells with and without flow. The modal coupling may also cause intensive energy exchange among these modes. Also, the presence of many co-existing modes can lead to a complex interaction between the non-linear modes leading to a rather complex non-linear behavior [33].

Expansions (13) have been thoroughly tested by Amabili and co-workers [23,25]. By comparison with experiments, they show that this modal expansion leads to both qualitatively and quantitatively correct results. In many papers the axi-symmetric deformation field is represented by a mode with twice the number of waves in the axial direction. It is readily obtained by perturbation procedures and is responsible for the main modal interaction and the softening behavior of the shell, as shown by the authors in [32]. However this mode has been traditionally represented by a Fourier expansion of odd sine functions as in Eq. (13) in order to satisfy the simply supported boundary conditions. When Eq. (13) is substituted into Eq. (7) even higher order interactions and the softening behavior of the shell, as shown by the authors in [32]. However this mode has been traditionally represented by a Fourier expansion of odd sine functions as in Eq. (13) in order to satisfy the simply supported boundary conditions. When Eq. (13) is substituted into Eq. (7) even higher harmonics emerge. If several modes with nearly equal frequencies are considered the relevant modal expansion must be derived as shown in [32].

The solution for the stress function may be written as \( F = F_h + F_p \), where \( F_h \) is the homogeneous solution and \( F_p \) the particular solution. The particular solution \( F_p \) is obtained analytically by substituting the assumed form of the lateral deflection on the left-hand side of the compatibility equation, Eq. (7), and by solving the resulting linear partial differential equation together with the relevant boundary and continuity conditions.

The homogeneous part of the stress function can be written as
\[
F_h = \frac{1}{2} \bar{N}_w x^2 \left( \bar{N}_{0,0} - \frac{1}{2 \pi R} \int_0^{2 \pi} \frac{\partial F}{\partial \alpha} R d\alpha d\chi \right) - \bar{N}_w R \theta,$
\]
(14)

where \( \bar{N}_w, \bar{N}_{0,0}, \) and \( \bar{N}_w \) are the average in-plane restraint stresses generated at the ends of the shell. This solution enables one to satisfy the in-plane boundary conditions on the average [25].

Boundary conditions allow us to express the in-plane restraint stresses \( \bar{N}_{xx}, \bar{N}_{0,0}, \) and \( \bar{N}_w \) in terms of \( w \) and its derivatives \[26,34\]
\[
\bar{N}_{xx} = N_{xx},
\]
(15)
\[
\bar{N}_{0,0} = A_{12} \frac{A_{11} - A_{22}}{A_{11}} \frac{1}{2 \pi} \int_0^{2 \pi} \left( \frac{W}{R} + \frac{1}{2} \left( \frac{\partial w}{\partial \alpha} \right)^2 \right) d\alpha d\chi,
\]
(16)
\[
\bar{N}_w = 0.
\]
(17)

Upon substituting the modal expressions for \( F \) and \( w \) into the equilibrium equation and applying the Galerkin method, a set of eight non-linear ordinary differential equations is obtained in terms of the time-dependent modal amplitudes, \( \xi(t) \).

2.3. Modeling of fluid–structure interaction

The Paidoussis and Denis [22] model is adopted to determine the perturbation pressure on the shell wall. In this model, linear potential theory is used to describe the effect of the internal axially flowing fluid. The fluid is assumed to be incompressible and inviscid and the flow to be isentropic and irrotational. The irrotationality property is the condition for the existence of a scalar potential function \( \Phi \), from which the velocity may be written as
\[
V = -\nabla \Phi.
\]
(18)

This scalar potential function is equal to \( \Phi = -U x + \Phi \), where the first term is associated with the undisturbed mean flow velocity \( U \), and the second term with shell motion. The potential function \( \Phi \) represents the velocity potential must satisfy the Laplace equation and the impenetrability condition at the shell–fluid interface, that is, \( \partial \Phi / \partial \alpha \big|_{r=R} = \partial \Phi / \partial \alpha + U \partial \Phi / \partial \chi \). Zero pressure is assumed at both ends of the contained fluid volume at \( x=0, L \). This corresponds to a long shell periodically supported by, for example, ring stiffeners, as usually found in pipelines.

Following previous studies and using the method of separation of variables [22,23,31], the perturbation pressure on the shell wall is found to be
\[
P_h = \rho_f L \pi \int_0^{2 \pi} \frac{\partial^2 w}{\partial \alpha^2} + 2 U \frac{\partial^2 w}{\partial \chi^2} + U^2 \frac{\partial^2 w}{\partial \chi^2},
\]
(19)

with
\[
f_w = \frac{1}{2 \pi} \int_0^{2 \pi} \frac{\partial \Phi}{\partial \alpha} R d\alpha d\chi,
\]
(20)

where \( \rho_f \) is the fluid density, \( I_n \) the \( n \)th order modified Bessel function and \( f_w \) its derivative with respect to its argument.

2.4. Linear analysis

Using the linearized equilibrium and compatibility equations, the natural frequency of the orthotropic axially loaded shell in the axial direction associated with the \( n \)th circumferential mode and with the \( m \)th axial mode is given by
The critical axial load is given by

\[
P_c = \left\{ \frac{D_{11}\left( \frac{m\pi}{L} \right)^2 + 2(D_{12} + 2D_{33}) \left( \frac{n}{R} \right)^4 + D_{22} \left( \frac{L}{m\pi} \right)^2}{R^2 \left\{ A_{11} \left( \frac{m\pi}{L} \right)^4 + A_{22} \left( \frac{n}{R} \right)^4 + \left[ (A_{11}A_{22} - A_{12}A_{33}) \left( \frac{n}{R} \right)^2 \left( \frac{L}{m\pi} \right)^2 \right] + \frac{1}{\bar{ho}_f \bar{w}^2 U^2} \right\}} \right\}
\]

and the critical flow velocity by

\[
U_{fc} = \frac{1}{\rho_f w} \left\{ \frac{D_{11}\left( \frac{m\pi}{L} \right)^3 + 2(D_{12} + 2D_{33}) \left( \frac{n}{R} \right)^2 + D_{22} \left( \frac{L}{m\pi} \right)^2}{R^2 \left\{ A_{11} \left( \frac{m\pi}{L} \right)^4 + A_{22} \left( \frac{n}{R} \right)^4 + \left[ (A_{11}A_{22} - A_{12}A_{33}) \left( \frac{n}{R} \right)^2 \left( \frac{L}{m\pi} \right)^2 \right] + \frac{1}{\bar{ho}_f \bar{w}^2 U^2} \right\}} \right\}
\]

3. Numerical results

A circular cylindrical shell with mass density \( \rho_s = 7850 \text{ kg/m}^3 \) and fluid density \( \rho_f = 1000 \text{ kg/m}^3 \) is considered in the numerical analysis. The damping coefficient is defined as \( c = 2\rho_f \omega_0 \), where \( \zeta \) is the viscous damping factor and \( \omega_0 \) the lowest natural frequency of the shell. In the present analysis, the adopted viscous damping factor is \( \zeta = 0.009 \). In the parametric analysis five different cases of orthotropic material [2, 10], with varying \( E_{00}/E_{xx} \) relations and Poisson ratios are adopted, while the reciprocal relation \( E_{xx}V_{00} = E_{00}V_{xx} \approx 0.274 \) remains constant. Case 1 has the lowest \( E_{00}/E_{xx} \) ratio and Case 5 has the highest one. Case 3 corresponds to the isotropic material and is used as benchmark for comparisons with other cases. Table 1 shows the shell material properties for each case.

In the parametric analysis, the following non-dimensional parameters are used for the axial coordinate, radial displacement, time, frequency, fluid-flow velocity, axial load and forcing frequency, respectively.

\[
\xi = \frac{x}{L}, \quad \omega = \frac{w}{\bar{w}}
\]

Table 1

<table>
<thead>
<tr>
<th>Case</th>
<th>( v_{xx} )</th>
<th>( v_{yy} )</th>
<th>( E_{xx} \times 10^{10} ) (N/m²)</th>
<th>( E_{yy} \times 10^{10} ) (N/m²)</th>
<th>( G_{xy} \times 10^{10} ) (N/m²)</th>
<th>( E_{00}/E_{xx} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.131926</td>
<td>0.012114</td>
<td>22.7350</td>
<td>2.0876</td>
<td>0.7958</td>
<td>0.0918</td>
</tr>
<tr>
<td>2</td>
<td>0.131926</td>
<td>0.012114</td>
<td>6.8599</td>
<td>2.0799</td>
<td>0.7958</td>
<td>1.0000</td>
</tr>
<tr>
<td>3</td>
<td>0.131926</td>
<td>0.012114</td>
<td>2.0545</td>
<td>2.0545</td>
<td>0.7958</td>
<td>2.3980</td>
</tr>
<tr>
<td>4</td>
<td>0.04</td>
<td>0.131926</td>
<td>2.0799</td>
<td>6.8599</td>
<td>0.7958</td>
<td>10.890</td>
</tr>
<tr>
<td>5</td>
<td>0.012114</td>
<td>0.131926</td>
<td>2.0876</td>
<td>22.7350</td>
<td>0.7958</td>
<td>10.890</td>
</tr>
</tbody>
</table>

Table 2

Comparison of natural frequency parameter (\( \Omega \)) for the orthotropic cylindrical shell with published results. \( L/R = 2.0, h/R = 0.01 \) and \( m = 1 \).

<table>
<thead>
<tr>
<th>Case</th>
<th>( \Omega )</th>
<th>( n )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
<th>( 7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present</td>
<td>1</td>
<td>0.129935</td>
<td>0.089316</td>
<td>0.067249</td>
<td>0.062844</td>
<td>0.052267</td>
<td>0.055545</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.226314</td>
<td>0.146659</td>
<td>0.104257</td>
<td>0.084419</td>
<td>0.081657</td>
<td>0.091799</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.365201</td>
<td>0.210309</td>
<td>0.140320</td>
<td>0.117922</td>
<td>0.127043</td>
<td>0.155192</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.376778</td>
<td>0.214806</td>
<td>0.150834</td>
<td>0.160873</td>
<td>0.202410</td>
<td>0.264719</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.381643</td>
<td>0.226661</td>
<td>0.201978</td>
<td>0.255783</td>
<td>0.360669</td>
<td>0.471173</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Warburton and Soni [2]</td>
<td>1</td>
<td>0.119875</td>
<td>0.085272</td>
<td>0.065184</td>
<td>0.054211</td>
<td>0.050975</td>
<td>0.054227</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.207798</td>
<td>0.139463</td>
<td>0.100747</td>
<td>0.081915</td>
<td>0.079310</td>
<td>0.089331</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.330057</td>
<td>0.198610</td>
<td>0.134684</td>
<td>0.113414</td>
<td>0.122501</td>
<td>0.150636</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.335754</td>
<td>0.201822</td>
<td>0.148874</td>
<td>0.153026</td>
<td>0.194244</td>
<td>0.256190</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.338104</td>
<td>0.210697</td>
<td>0.182208</td>
<td>0.240929</td>
<td>0.334672</td>
<td>0.453548</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Li and Chen [10]</td>
<td>1</td>
<td>0.119888</td>
<td>0.085277</td>
<td>0.065190</td>
<td>0.054336</td>
<td>0.051014</td>
<td>0.054286</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.207822</td>
<td>0.139494</td>
<td>0.100772</td>
<td>0.081952</td>
<td>0.079335</td>
<td>0.089433</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.330100</td>
<td>0.198607</td>
<td>0.134742</td>
<td>0.113515</td>
<td>0.122652</td>
<td>0.150753</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.335802</td>
<td>0.201833</td>
<td>0.148975</td>
<td>0.153209</td>
<td>0.194549</td>
<td>0.256795</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.338120</td>
<td>0.210729</td>
<td>0.188495</td>
<td>0.241727</td>
<td>0.336395</td>
<td>0.456826</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[ \tau = \lambda t, \text{ with } \lambda = \frac{E_o}{\rho R^2(1 - \nu_{xx} \nu_{yy})}. \]  

(25)

\[ \Omega = \frac{\omega}{\lambda}. \]  

(26)

\[ U_b = \frac{U}{V_e}, \text{ with } V_e = \frac{\pi^2}{12\rho / (1 - \nu_{xx} \nu_{yy})}. \]  

(27)

Table 3

<table>
<thead>
<tr>
<th>Case</th>
<th>n</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.112433</td>
<td>0.099815</td>
<td>0.094012</td>
<td>0.094277</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.187830</td>
<td>0.162757</td>
<td>0.151993</td>
<td>0.153989</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.289210</td>
<td>0.245100</td>
<td>0.233375</td>
<td>0.247519</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.312636</td>
<td>0.292523</td>
<td>0.317612</td>
<td>0.374474</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.371037</td>
<td>0.409394</td>
<td>0.502792</td>
<td>0.632017</td>
<td></td>
</tr>
</tbody>
</table>

Table 4

<table>
<thead>
<tr>
<th>Case</th>
<th>n</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0628</td>
<td>0.0585</td>
<td>0.0575</td>
<td>0.0597</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.1049</td>
<td>0.0954</td>
<td>0.0929</td>
<td>0.0975</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.1616</td>
<td>0.1437</td>
<td>0.1426</td>
<td>0.1568</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.1746</td>
<td>0.1715</td>
<td>0.1941</td>
<td>0.2372</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.2073</td>
<td>0.2401</td>
<td>0.3073</td>
<td>0.4003</td>
<td></td>
</tr>
</tbody>
</table>

Table 5

<table>
<thead>
<tr>
<th>Case</th>
<th>(m,n)</th>
<th>( \Omega_0 )</th>
<th>( U_{bc} )</th>
<th>( \Gamma_{bc} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,7)</td>
<td>0.0575</td>
<td>0.8111</td>
<td>0.1556</td>
</tr>
<tr>
<td>2</td>
<td>(1,7)</td>
<td>0.0929</td>
<td>1.3113</td>
<td>0.4147</td>
</tr>
<tr>
<td>3</td>
<td>(1,7)</td>
<td>0.1426</td>
<td>2.0135</td>
<td>0.9387</td>
</tr>
<tr>
<td>4</td>
<td>(1,6)</td>
<td>0.1715</td>
<td>2.3659</td>
<td>1.5246</td>
</tr>
<tr>
<td>5</td>
<td>(1,5)</td>
<td>0.2073</td>
<td>2.7919</td>
<td>2.4212</td>
</tr>
</tbody>
</table>

To validate the mathematical model, Table 2 shows the comparison of the natural frequency parameter \( \Omega \) of an empty orthotropic cylindrical shell, obtained in this work using Eq. (21) for different values of \( n \) with those obtained by Warburton and Soni [2] and by Li and Chen [10] for \( L/R = 2.0, h/R = 0.01 \), and \( m = 1 \), using Flügge’s theory and considering in-plane inertia effects, which are neglected in the present formulation. For values of \( n \approx 5 \) there is a palpable difference due to the different shell theories used in the three analyses. However, for \( n \geq 5 \) the values are very close, as expected. The frequency parameter used in Table 2 is given by [10]

\[ \Xi^2 = \frac{\rho R^2 \Omega_0^2 (1 - \nu_{xx} \nu_{yy})}{E_{xx}}. \]  

(30)

The lowest natural frequency parameter increases as the \( E_{xx}/E_{xx} \) ratio increases, while the circumferential wave-number \( n \) corresponding to the lowest frequency decreases.

In the parametric analysis, a cylindrical shell with \( L/R = 1.0 \) and \( h/R = 0.01 \) is considered. The variation of the natural frequency for the empty shell with the circumferential wave-number \( n \) and \( m = 1 \), considering the five different orthotropic shell materials presented in Table 1, is shown in Table 3, while Table 4 presents the same results for the fluid-filled shell. The decrease in the natural frequency due to the internal fluid does not depend on the shell material properties but only on the circumferential wave-number \( n \). As \( n \) increases, the ratio between the natural frequency of the fluid filled and the empty shell \( (\Omega_0/\Omega_e) \) increases due to the decreasing effect of the added mass of the fluid on the shell frequencies. For example, for \( n = 5 \), \( \Omega_0/\Omega_e = 0.56 \), while for \( n = 8 \), \( \Omega_0/\Omega_e = 0.63 \), independent of the shell material. The lowest natural frequency, \( \Omega_0 \), critical fluid-flow velocity, \( U_{bc} \), and axial buckling load, \( \Gamma_{bc} = P_{c} / P_{cr} \), (see Eqs. (21)–(23)) of each shell as well as the wave-numbers \( (m,n) \) associated with the minimum value are shown in Table 5. These values are used as reference values in the following non-linear analysis. The lowest natural frequency for all cases is associated with a value of \( n \approx 5 \), so the non-linear analysis of the fluid-filled shell can be performed with reasonable accuracy using Donnell’s non-linear shallow-shell theory.

Fig. 2 shows the post critical paths for a fluid-filled shell without flow under uniform axial compression, considering the
Fig. 3. Non-linear frequency–amplitude relations for orthotropic empty shell and varying $n$. $U_b = 0.0$ and $f_0 = 0.0$. (a) Case 1, (b) Case 2, (c) Case 4, and (d) Case 5. $n = 5$; $n = 6$; $n = 7$; and $n = 8$.

Fig. 4. Non-linear frequency–amplitude relations for orthotropic fluid-filled shell and varying $n$. $U_b = 0.0$ and $f_0 = 0.0$. (a) Case 1, (b) Case 2, (c) Case 4 and (d) Case 5. $n = 5$; $n = 6$; $n = 7$; and $n = 8$. 
four different orthotropic shell materials presented in Table 1. As shown in Fig. 2(a), as the ratio $E_{yy}/E_{xx}$ increases, the critical load increases, and the non-linear post-buckling behavior of the shell changes radically. For comparison purposes, Fig. 2(b) shows the post-buckling paths considering a normalized critical load, that is, for each path the load parameter is divided by the associated critical load, $\Gamma_{cr}$, shown in Table 5. For $E_{yy}/E_{xx}=0.0918$, Case 1, the non-linearity is very small. As the ratio $E_{yy}/E_{xx}$ increases, the initial curvature of the unstable post-buckling path increases and the difference between the critical load, $\Gamma_{cr}$, and the folding point corresponding to the minimum post-critical load, $\Gamma_{min}$, also increases. The difference $(\Gamma_{cr}-\Gamma_{min})/\Gamma_{cr}$ increases as follows (see Fig. 2(b)): for Case 1: 0.17, Case 2: 0.54, Case 4: 0.63, and Case 5: 0.63. This means that the imperfection sensitivity of the orthotropic shell increases in a similar way, being the minimum post-critical load, $\Gamma_{min}$, a lower bound of the limit load of the imperfect structure when the membrane stiffness has been completely eroded by the imperfections [35]. Also, as the ratio $E_{yy}/E_{xx}$ increases, the lateral displacements of the shell in the post-buckling range increase. This has a strong influence on the

---

**Fig. 5.** Comparisons of non-linear frequency–amplitude relations for the same $n$ value: (a) $n=5$, (b) $n=6$, (c) $n=7$, and (d) $n=8$. ———, Case 1; ———, Case 2; ———, Case 4; ———, and Case 5.

---

**Fig. 6.** Non-linear frequency–amplitude relations of the fluid filled shell associated with the lowest natural frequency of each shell ($\Omega$). ———, Case 1; ———, Case 2; ———, Case 4; ———, and Case 5 (a) Non-normalized frequency, (b) normalized frequency.
Fig. 7. Influence of the fluid-flow velocity on the post-buckling behavior of the fluid-filled shell: (a) Case 1, (b) Case 2, (c) Case 4, and (d) Case 5. Black line: $U_b = 0.0$; gray line: $U_b = 0.20U_{bcr}$; dotted line: $U_b = 0.40U_{bcr}$.

Fig. 8. Influence of the fluid-flow velocity on the non-linear frequency–amplitude relation of the fluid-filled shell: (a) Case 1, (b) Case 2, (c) Case 4, and (d) Case 5. Black line: $U_b = 0.0$; gray line: $U_b = 0.20U_{bcr}$; dotted line: $U_b = 0.40U_{bcr}$. 
non-linear dynamics of the pre-loaded shell as shown later in the present work.

Fig. 3 shows the non-linear frequency–amplitude relations, for varying $n$, for the empty orthotropic cylindrical shell, while Fig. 4 shows the same results considering a fluid-filled shell with no flow and no static pre-load.

For all values of $n$ the shell displays a softening behavior, but the level of non-linearity depends on the circumferential wave-number, $n$. Clearly, the softening effect increases with $n$. The number of circumferential waves also affects the point where the bending back of the backbone curve (turning point) associated with large bending effects occurs. All curves intersect, showing that, for certain vibration amplitudes, internal (1:1) resonance associated with different neighboring modes may occur.

In Fig. 3(a) and (b) the non-linear frequency–amplitude relations for, respectively, Case 1 and Case 2 are plotted. The level of non-linearity increases with the circumferential wave-number, and the lowest bending back point is associated with the lowest natural frequency.

Fig. 3(c) shows the frequency–amplitude relations for Case 4. Again, as in Fig. 3(a) and (b), the level of non-linearity is associated to the number of circumferential waves but in this case, the lowest turning point does not correspond to the lowest natural frequency parameter. Finally, in Fig. 3(d) the

---

**Fig. 9.** Instability boundaries for a perfect shell with no static pre-load ($F_0 = 0.0$). (a), (b): Fluid-filled shell with $U_b = 0.0$; (c), and (d): fluid-filled shell with $U_b = 0.30U_{cr}$.

**Fig. 10.** Instability boundaries for a perfect empty shell with no static pre-load ($F_0 = 0.0$). Green color: Case 1; gray color: Case 2; red color: Case 3; black color: Case 4; and blue color: Case 5. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
frequency–amplitude relations for Case 5 are displayed. In this case, the plots clearly show the non-linearity associated to the number of circumferential waves and there are no curve intersections up to the maximum vibration amplitude analyzed here (six times the shell thickness). The non-linear behavior observed in Fig. 4 for the fluid-filled shell is similar to that of the empty shell, but the degree of non-linearity is higher and the frequency–amplitude relations may have up to two crossings, as observed in Fig. 4(a) and (b). In all cases depicted in Fig. 3 and 4 the modal amplitude $\xi_{1,1}$ at which the turning point occurs increases with $E_{yy}/E_{xx}$ and in each case it decreases with $n$. The initial softening behavior is associated with the loss of membrane stiffness, particularly in the circumferential direction, while the stiffening effect observed for large amplitude vibrations is related with increasing bending stiffness.

To study the influence of the $E_{yy}/E_{xx}$ ratio on the frequency–amplitude relation, Fig. 5 shows the comparison of the non-linear frequency–amplitude relations of each studied case for the same value of circumferential wave-number $n$. As in Fig. 4, a perfect fluid-filled shell without flow and static pre-load is considered in the analysis. The results show that the non-linearity increases with the $E_{yy}/E_{xx}$ ratio, in a fashion similar to that observed in the post-buckling relation.

Fig. 6(a) and (b) shows the non-linear frequency–amplitude relation corresponding to the lowest natural frequency for the four different cases of material orthotropy analyzed here. In Fig. 6(b) the frequency $\Omega$ is normalized by lowest natural frequency of each shell, $\Omega_0$. The results confirm the influence of the $E_{yy}/E_{xx}$ ratio on the degree of softening. These backbone curves are particularly important for the analysis and understanding of

Fig. 11. Bifurcation diagrams of the Poincaré map for an empty shell with $\psi=0.8(2\Delta_0)$ and $\Gamma_0=0.0$. Variation of the generalized coordinate of the driven mode as a function of the dynamic axial load. (a) Case 1, (b) Case 2, (c) Case 3, (d) Case 4 and (e) Case 5.
the forced vibrations and bifurcations of the shell under lateral and axial harmonic excitation in the spectral neighborhood of the lowest resonances. It is interesting to observe that while the $E_{yy}/E_{xx}$ ratio increases the critical load and natural frequency, at the same time it increases the softening effect of the response. This means that the increase in stiffness that leads to higher frequencies and critical loads is eroded by the non-linearity as the lateral deflections increase. From the results of Figs. 3–6, one concludes that in the spectral neighborhood of the lowest resonances the forced damped shell displays predominantly softening resonance curves as will be shown herein.

Fig. 7 illustrates the influence of the fluid-flow velocity on the post-buckling behavior of the shells under uniform axial compression. The critical load decreases as the fluid velocity increases. However for each orthotropic material the shell displays the same post-buckling behavior; that is, the flow velocity in the interval investigated here does not change the degree or type of non-linearity of the shell. Fig. 8 shows the influence of the fluid-flow velocity on the non-linear frequency–amplitude relation. Independent of the flow velocity, the shell displays the same softening behavior, with a slight enhancement as $U_b$ increases.
Now consider the shell under dynamic axial load, see Eqs. (12) and (28) ($\Gamma_1 \neq 0$). Fig. 9 shows the instability boundaries for a slowly evolving system in the excitation frequency–amplitude control space, considering the five cases shown in Table 1. For each value of excitation frequency parameter $\psi$, Eq. (29), the non-dimensional forcing amplitude $\Gamma_1$ is increased slowly until parametric instability occurs. The parametric instability boundary is the limit where small perturbations from the trivial solution will result in an initial exponential growth in the oscillations. The parametric instability boundary is composed of various curves, each one associated with a particular bifurcation event. The first important instability region is associated with the direct resonance zone when the frequency of excitation is equal to the lowest natural frequency of the fluid-filled shell ($\psi = \Omega_0$). The second well to the right is associated to the principal parametric instability region and occurs when the frequency of excitation is equal to two times the lowest natural frequency of the fluid-filled shell ($\psi = 2\Omega_0$). The value of the lowest natural frequency of the fluid-filled shell for each case is given in Table 5.

Fig. 13. Bifurcation diagrams of the Poincaré map for an empty shell with $\psi = 1.0(2\Omega_0)$ and $\Gamma_0 = 0.0$. Variation of the generalized coordinate of the driven mode as a function of the dynamic axial load. (a) Case 1, (b) Case 2, (c) Case 3, (d) Case 4 and (e) Case 5.
$U_{bc}$ is the critical flow velocity. In Fig. 9(a) and (c) we compare the instability boundaries of Cases 1, 2, and 3 (isotropic case). The instability boundary of Case 3 (isotropic material) is compared with those of Case 4 and 5 in Fig. 9(b) and (d). In all cases the upper region of the instability boundary, between the two resonance regions shows a kind of fractal boundary. The variation of the parametric instability load increases as the $E_{yy}/E_{xx}$ ratio and the non-linearity of the structure increase. While the descending left-hand branch of each region, associated with sub-critical bifurcations [27,31], exhibits a smooth boundary, the ascending right hand branch, associated with super-critical bifurcations [27,31], show a complex boundary with various discontinuities possibly due to secondary bifurcation events. As the $E_{yy}/E_{xx}$ ratio increases, the stability boundaries move to the higher frequency region and the critical load parameter increases, as illustrated in Fig. 9 for a fluid-filled shell with $U_b=0.0$ (Fig. 9(a) and (b)) and $U_b=0.30U_{bc}$. When comparing the boundaries of the system with and without flow, Fig. 9(a–b) and (c–d), it is possible to observe that the fluid velocity has the effect of shifting all the instability boundaries to the left. This is mainly due to the fact that the natural frequencies decrease with increasing flow velocity. The flow velocity may increase or decrease the critical load, depending on the value of the forcing frequency. However, if the forcing frequency is divided by the natural frequency of each shell material and the forcing amplitude is divided by the critical load of each shell (see Table 5), the five

![Bifurcation diagrams for different cases](image)

**Fig. 14.** Bifurcation diagrams of the Poincaré map for an empty shell with $\psi=1.1(2\Omega_0)$ and $\Gamma_0=0.0$. Variation of the generalized coordinate of the driven mode as a function of the dynamic axial load. (a) Case 1, (b) Case 2, (c) Case 3, (d) Case 4 and (e) Case 5.
instability boundaries are practically coincident, as shown in Fig. 10 for an empty shell. Differences only occur between the two main instability regions associated with the lowest natural frequency and the one around twice this value. Near the two reference frequencies the curves are coincident; this means that the Strutt diagram in each region is governed by the same non-dimensional Mathieu–Hill equations, independent of the shell material. A similar figure is obtained for the fluid-filled shell with or without flow.

However, after parametric bifurcation occurs, the sequence of bifurcation events is highly dependent on the shell material, as illustrated in Figs. 11–15 where the bifurcation diagrams for each material are plotted as a function of the excitation amplitude for the same non-dimensional frequency ratio. The five selected values of the excitation frequency $\psi$ are: $0.8(2\Omega_0)$, $0.9(2\Omega_0)$, $2\Omega_0$, $1.1(2\Omega_0)$, and $1.2(2\Omega_0)$, where $\Omega_0$ is the lowest natural frequency of each material given in Table 5. That is, two values before, two after and one value corresponding to the minimum parametric instability load in the main parametric resonance region. These bifurcation diagrams are obtained by the brute-force method, by increasing the forcing amplitude $G_1$. By analyzing the results, one can conclude that the shell material does not change the type of parametric bifurcation associated with the instability boundary in the main parametric resonance region: for $\psi < 2\Omega_0$, the shell

![Bifurcation diagrams](image-url)
displays a subcritical bifurcation while for $\psi > 2\Omega_0$ the bifurcation is supercritical. The sub- and supercritical bifurcations connected with, respectively, the left and right sides of each resonance region are a characteristic of softening systems as those analyzed here. For $\psi < 2\Omega_0$ (Figs. 11 and 12), after the critical point, the trivial solution becomes unstable and the system jumps to either a period-two or chaotic large-amplitude oscillation, depending on the shell material. Also the subsequent sequence of bifurcations is highly dependent on the shell material and excitation frequency. As a rule, the amplitude of the post-critical oscillation after the parametric instability increases as the $E_{yy}/E_{xx}$ ratio increases. At $\psi < 2\Omega_0$ (Fig. 13), there is the transition from sub- to supercritical bifurcation. At the bifurcation point the trivial solution becomes unstable and the shell displays a small-amplitude period two solution, with a sudden increase in the vibration amplitude just after the bifurcation point. As $\psi$ increases beyond this value, see, for example, Fig. 14, the initial growth of the bifurcated period two solution becomes more gradual. However, for larger values of the excitation frequency, as shown in Fig. 15 for $\psi = 1.2(2\Omega_0)$ after bifurcation the shell may display either a small amplitude period two solution or large amplitude chaotic motions. Similar to the static case, Figs. 11–15 show that the critical parametric

Fig. 16. Maximum amplitude of vibration of the first asymmetric mode (driven mode) versus the excitation frequency for $F_1=0.01$ and $U_b=0.00$. (a) Case 1, (b) Case 2, (c) Case 3, (d) Case 4 and (e) Case 5. Dotted lines: unstable points; and continuous lines: stable points.
instability load increases with the $E_{yy}/E_{xx}$ ratio. The main parametric instability region around $\psi = 2\Omega_0$ is the most dangerous instability region and the dynamic critical load $G_{1cr}$ is just a fraction of the static critical load $G_{0cr}$, as one can observe, for example, in Fig. 13 where this decrease is of about 50%.

Now the behavior of the orthotropic shell under a harmonic lateral pressure described by

$$f = f_n \sin(m\pi \xi) \cos(n\pi \eta) \cos(\varphi \varepsilon),$$

$$f_n = F_l h^2 \rho_s \omega_o^2,$$

$$\varphi = \omega / \lambda.$$

is considered, where $\omega_o$ and $\varphi$ are the dimensional and non-dimensional frequencies of lateral pressure, respectively.

Figs. 16–18 show the resonance curves for the driven mode as a function of the forcing parameter $\varphi / \Omega_0$ considering the forcing amplitude $F_l = 0.01$ with no axial load and an increasing value for the flow velocity, namely, $U_b = 0.00$, $U_b = 0.20U_{bcr}$, and $U_b = 0.40U_{bcr}$. These figures have been obtained by continuation techniques. Continuous and dotted lines correspond, respectively, to stable and unstable solutions. The results show that the shell material has a strong influence on the resonance curves. As the $E_{yy}/E_{xx}$ ratio increases, the non-linearity decreases, as well as the

Fig. 17. Maximum amplitude of vibration of the first asymmetric mode (driven mode) versus the excitation frequency for $F_l = 0.01$ and $U_b = 0.20U_{bcr}$. (a) Case 1, (b) Case 2, (c) Case 3, (d) Case 4 and (e) Case 5. Dotted lines: unstable points; and continuous lines: stable points.
range where unstable solutions occur; for $E_{yy}/E_{xx} = 10.890$ (Case 5), the response becomes practically linear. However, there is a slight increase of the non-linearity as the fluid-flow velocity $U_b$ increases. Compare, for example, Figs. 16(e), 17(e) and 18(e), corresponding to Case 5. As $U_b$ increases, the resonance curves increasingly bend to the left, increasing the unstable region between the two folding points. The same occurs in the other cases, and the non-linear behavior becomes more involved as $E_{yy}/E_{xx}$ decreases. The variation of the amplitude of the companion mode amplitude for each fluid-flow velocity is depicted in Figs. 19–21 as a function of the forcing parameter $\phi/\Omega_0$. Independent of the value of the $U_b$, the companion mode is not excited in Case 5. For low frequency values, the non-resonant branch of the resonance curve is always stable up to the saddle-node bifurcation where the curve bends back. On the other hand, excepting Case 5, the resonant branch becomes unstable, as the forcing frequency decreases due to a pitchfork bifurcation [23] and simultaneously the companion mode is excited and participates in the oscillations with nonzero value in the main resonance region around $\phi/\Omega_0 \approx 1$. Fig. 16(f) shows a bifurcation diagram obtained by the brute-force method for Case 1 (compare with Fig. 16(a)). Here we observe that there is a
Fig. 19. Maximum amplitude of the first asymmetric companion mode versus the excitation frequency for $F_L=0.01$ and $U_b=0.00$. (a) Case 1, (b) Case 2, (c) Case 3 and (d) Case 4. Dotted lines: unstable points; and continuous lines: stable points.

Fig. 20. Maximum amplitude of the first asymmetric companion mode versus the excitation frequency for $F_L=0.01$ and $U_b=0.20U_{bc}$. (a) Case 1, (b) Case 2, (c) Case 3 and (d) Case 4. Dotted lines: unstable points; and continuous lines: stable points.
region where no stable periodic solution occurs and the shell exhibits chaotic motion. In all Cases this corresponds to the region where both the driven and companion modes are unstable (see Figs. 16–21).

4. Conclusions

In this work, the non-linear vibrations of a perfect simply supported fluid-filled orthotropic cylindrical shell subjected to time-dependent axial and lateral pressure loads are analyzed. To model the shell, the Donnell non-linear shallow-shell theory without considering the effect of shear deformation is used while the fluid is assumed to be incompressible and inviscid and the flow to be isentropic and irrotational.

Four different orthotropic materials are analyzed. Results show that the material orthotropy has a strong influence on the critical loads, natural frequencies, post-buckling paths and frequency–amplitude relations. The non-linearity increases as the ratio of Young’s moduli in the circumferential and axial direction, $E_{yy}/E_{xx}$, increases.

When a harmonic axial load is applied, the parametric instability boundaries are also influenced by the shell orthotropy. The variation of the critical load as a function of the forcing frequency increases with $E_{yy}/E_{xx}$ and the instability curves are shifted to a higher frequency range. On the other hand, the fluid velocity has the effect of shifting all the instability boundaries to the left. The flow velocity may increase or decrease the critical load, depending on the value of the forcing frequency.

When a radial pressure is applied, the shell material has a strong influence on the resonance curves, for low $E_{yy}/E_{xx}$ ratio the shell exhibits complex non-linear behavior and, as the $E_{yy}/E_{xx}$ ratio is increased, the non-linearity decreases as well as the range where unstable solutions occur.

Acknowledgments

This work was made possible by the support of the Brazilian Ministry of Education—CAPES and CNPq. The support by NSERC of Canada is also gratefully acknowledged. The second author acknowledges the support by FAPERJ-CNE.

References


