Shear Stress in Beams and Thin-walled Members

Shear on the Horizontal Face of a Beam Element

- Consider prismatic beam
- For equilibrium of beam element
  \[ \sum F_y = 0 = \Delta H + \int (\sigma_y - \sigma_n) \, dA \]
  \[ \Delta H = \frac{M_D - M_C}{I} \int_y dA \]
- Note,
  \[ Q = \int_y dA \]
  \[ M_D - M_C = \frac{dM}{dx} \Delta x = V \Delta x \]
- Substituting,
  \[ \Delta H = \frac{VQ}{I} \Delta x \]
  \[ q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = \text{shear flow} \]
Shear on the Horizontal Face of a Beam Element

- Shear flow,
  \[ q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = \text{shear flow} \]
- where
  \[ Q = \int \gamma dA \]
  = first moment of area above \( \gamma \)
  \[ I = \int \gamma^2 dA \]
  = second moment of full cross section

Example 6.01

Knowing that the spacing between nails is 25 mm and that the vertical shear in the beam is \( V = 500 \text{ N} \), determine the shear force in each nail.
Example 6.01

**SOLUTION:**

\[ Q = Ay \]
\[ = (0.020 \text{ m} \times 0.100 \text{ m})(0.060 \text{ m}) \]
\[ = 120 \times 10^{-6} \text{ m}^3 \]
\[ I = \frac{1}{12} (0.020 \text{ m})(0.100 \text{ m})^3 \]
\[ + 2\left[ \frac{1}{12} (0.100 \text{ m})(0.020 \text{ m})^3 \right] \]
\[ + (0.020 \text{ m} \times 0.100 \text{ m})(0.060 \text{ m})^2 \]
\[ = 16.20 \times 10^{-6} \text{ m}^4 \]

\[ q = \frac{VQ}{I} = \frac{(500 \text{ N})(120 \times 10^{-6} \text{ m}^3)}{16.20 \times 10^{-6} \text{ m}^4} \]
\[ = 3704 \text{ N/m} \]

- shear force in each nail:
\[ F = (0.025 \text{ m})q = (0.025 \text{ m})(3704 \text{ N/m}) \]
\[ F = 92.6 \text{ N} \]

**Determination of the Shearing Stress in a Beam**

- Shear flow,
\[ q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = \text{shear flow} \]

- The **average** shearing:
\[ \tau_{\text{ave}} = \frac{q}{b} = \frac{VQ}{Ib} \]
Shearing Stresses $\tau_{xy}$ in Common Types of Beams

- For a narrow rectangular beam,

$$\tau_{xy} = \frac{VQ}{Ib} = \frac{3V}{2A} \left(1 - \frac{c^2}{b^2}\right)$$

$$\tau_{\text{max}} = \frac{3V}{2A}$$

- For I beams

$$\tau_{\text{ave}} = \frac{VQ}{It}$$

$$\tau_{\text{max}} = \frac{V}{A_{web}}$$

Sample Problem 6.2

Knowing that for the above timber beam,

$$\sigma_{\text{ult}} = 1800 \text{ psi} \quad \tau_{\text{ult}} = 120 \text{ psi}$$

determine the minimum required depth $d$ of the beam.
Sample Problem 6.2

SOLUTION:

\[ V_{\text{max}} = 3 \text{kips} \]
\[ M_{\text{max}} = 7.5 \text{kip} \cdot \text{ft} = 90 \text{kip} \cdot \text{in} \]

\[ \sigma_{\text{all}} = \frac{M_{\text{max}}}{S} \]
\[ 1800 \text{ psi} = \frac{90 \times 10^3 \text{lb} \cdot \text{in.}}{(0.5833 \text{ in.})d^2} \]
\[ d = 9.26 \text{ in.} \]

\[ \tau_{\text{all}} = \frac{3V_{\text{max}}}{2A} \]
\[ 120 \text{ psi} = \frac{3 \times 3000 \text{lb}}{2(3.5 \text{ in.})d} \]
\[ d = 10.71 \text{ in.} \]
Shearing Stresses in Thin-Walled Members

\[ \Delta H = \frac{\Delta M}{I} Q \]

- The corresponding shear stress is
  \[ \tau_{xy} = \tau_{xz} \approx \frac{\Delta H}{t \Delta x} = \frac{VQ}{It} \]

- Previously, for the shearing stress in the web
  \[ \tau_{sy} = \frac{VQ}{It} \]

- NOTE: \( \tau_{xy} \approx 0 \) in the flanges
  \( \tau_{xz} \approx 0 \) in the web

\[ q = \frac{VQ}{I} \]

\[ \tau = \frac{VQ}{I(2t)} \]
Shearing Stresses in Thin-Walled Members

- The continuity of the variation in \( q \) suggests an analogy to fluid flow.

Sample Problem 6.3

Knowing that the vertical shear is 50 kips in a steel beam, determine the horizontal shearing stress in the top flange at the point \( a \).

\[ I = 394 \text{ in}^4 \]

SOLUTION:

- For the shaded area,
  \[ Q = (4.31\text{ in})(0.770\text{ in})(4.815\text{ in}) \]
  \[ = 15.98\text{ in}^3 \]

- The shear stress at \( a \),
  \[ \tau = \frac{VQ}{It} = \frac{50\text{ kips}(15.98\text{ in}^3)}{394\text{ in}^4(0.770\text{ in})} \]
  \[ \tau = 2.63\text{ ksf} \]
Unsymmetric Loading of Thin-Walled Members

- Beam without twisting.

\[ \sigma_x = -\frac{My}{I} \quad \tau_{ave} = \frac{VQ}{It} \]

- Beam twists under loading.

\[ \sigma_x = -\frac{My}{I} \quad \tau_{ave} \neq \frac{VQ}{It} \]

- When the force P is applied at a distance e to the left of the web centerline, the member bends in a vertical plane without twisting.

\[ Fh = Ve \]

\[ \tau_{ave} = \frac{VQ}{It} \quad V = \frac{D}{B} q \, ds \quad F = \frac{B}{A} q \, ds = -\frac{E}{D} q \, ds = -F' \]
Example 6.05

- Determine the location for the shear center of the channel section with \( b = 4 \) in., \( h = 6 \) in., and \( t = 0.15 \) in.

\[
e = \frac{Fh}{V}
\]

where

\[
F = \int_0^b \frac{VQ}{I} ds = \frac{Vb}{I} \int_0^h \frac{h}{2} ds = \frac{Vbh^2}{4I}
\]

\[
I = I_{\text{web}} + 2I_{\text{flange}} = \frac{1}{12}th^3 + 2 \left[ \frac{1}{12}th^3 + bh\left(\frac{h}{2}\right)^2 \right]
\]

Combining,

\[
e = \frac{b}{2 + \frac{h}{3b}} = \frac{4\text{in.}}{2 + \frac{6\text{in.}}{3(4\text{in.})}} = 1.6\text{in.}
\]

Example 6.06

- Determine the shear stress distribution for \( V = 2.5 \) kips.

\[
\tau = \frac{q}{t} = \frac{VQ}{It}
\]

- Shearing stresses in the flanges,

\[
\tau_f = \frac{VQ}{It} = \frac{V}{2} \left(\frac{h}{2}\right) = \frac{V}{2} \frac{h}{2} = \frac{6Vb}{2(1/2)th^3(6b + h)} = \frac{6(2.5\text{kips})(4\text{in.})}{(0.15\text{in})(6\text{in})(6 \times 4\text{in} + 6\text{in})} = 2.22\text{ksi}
\]

- Shearing stress in the web,

\[
\tau_{\text{max}} = \frac{VQ}{It} = \frac{V(\frac{1}{2}ht)(4b + h)}{1/2 th^2(6b + h)} = \frac{3(2.5\text{kips})(4 \times 4\text{in} + 6\text{in})}{2(0.15\text{in})(6\text{in})(6 \times 6\text{in} + 6\text{in})} = 3.06\text{ksi}
\]